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Wavefunctions of a free electron in an external field and their application in intense field interactions:

I. Non-relativistic treatment

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Abstract. The behaviour of a free electron in a homogeneous but time varying external field is analysed and exact results are presented. Based on the exact wavefunction obtained, a new perturbation method for treating intense field problems is proposed. In particular, the transition amplitude of nonlinear direct and inverse bremsstrahlung is calculated.

1. Introduction

A detailed analysis is given here of the behaviour of a free electron in a homogeneous external field. We consider separately the case of the constant and the periodically time-dependent fields. For the description of the electron the Schrödinger equation is used in a non-relativistic treatment together with the dipole, or long wavelength, approximation of the field. The general solution of the problem is given and it is shown how it can be matched to different initial conditions. By choosing special initial conditions the stationary solution in a constant field (Landau and Lifshitz 1963) and the plane wave solution in a periodic field (Keldysh 1965) are reobtained. By using this last set of solutions we develop a perturbation method for treating intense field problems and we give the expression for multiphoton free-free transition matrix elements (nonlinear inverse bremsstrahlung). The relationship between this method and other approximation methods (Henneberger 1968, Faisal 1973) is also established and we give the expression for multiphoton transition matrix elements as well. The transition matrix element has the form predicted previously (Bergou 1975) for the case of a periodic Hamiltonian.

2. Electron in a constant electric field

The Schrödinger equation for an electron in a homogeneous constant electric field with amplitude \mathbf{E}_0 is

$$(\hat{\mathbf{p}}^2/2m - e\mathbf{E}_0\hat{\mathbf{x}})\psi = i\hbar \partial\psi/\partial t. \quad (1)$$

Operators are denoted by $\hat{}$. m is the mass of the electron, e its charge and \hbar Planck's constant divided by 2π . Vector quantities are denoted by bold type.

As the Hamiltonian does not depend on time, we can look for the stationary solution in the form $\psi(\mathbf{x}, t) = \exp[-(i/\hbar)Et]u(\mathbf{x})$. Instead of solving the corresponding equation for $u(\mathbf{x})$ we perform the following gauge transformation:

$$\Phi' = \Phi - \frac{1}{c} \frac{\partial \chi}{\partial t}, \quad \mathbf{A}' = \text{grad } \chi, \quad \chi = -c\mathbf{E}_0 \mathbf{x}t.$$

Here Φ and \mathbf{A} are the scalar and vector potentials, respectively, and χ is chosen so that in the new gauge $\Phi' = 0$. If, furthermore, in the new gauge $\psi' = \exp[(ie/\hbar c)\chi]\psi$ (Schiff 1955), then ψ' satisfies the following wave equation:

$$\frac{1}{2m} \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A}' \right)^2 \psi' = i\hbar \frac{\partial \psi'}{\partial t}. \quad (1a)$$

As the transformation between ψ and ψ' is unitary, (1) and (1a) give completely equivalent descriptions of the same problem. Nevertheless, (1a) is more convenient for practical calculation because in the momentum representation the Hamiltonian becomes diagonal and the equation is readily integrable. Its general solution is

$$\psi'(\mathbf{p}, t) = f(\mathbf{p}) \exp\left(-\frac{i}{\hbar} \frac{1}{2m} \int_0^t (\mathbf{p} + e\mathbf{E}_0\tau)^2 d\tau\right). \quad (2)$$

Here $f(\mathbf{p})$ is an as yet unspecified function of the momentum, to be determined from the initial, boundary, or any subsidiary conditions (such as completeness and orthonormality). It is interesting to note at this point that in contrast to (1), equation (1a) has no stationary solution because its Hamiltonian explicitly depends on time. The stationary solution to (1) corresponds to a special choice of $f(\mathbf{p})$ in (2), namely (in one dimension)

$$f_E(p) = \frac{1}{(2\pi\hbar eE_0)^{1/2}} \exp\left[-\frac{i}{\hbar} \left(\frac{p^3}{6meE_0} - \frac{Ep}{eE_0} \right)\right]. \quad (3)$$

Here E is the energy of the stationary state (the separation constant of equation (1)) and with this choice of the normalisation constant the states are properly orthonormalised on an energy scale (Landau and Lifshitz 1963). The present approach makes use of a peculiarity of the problem, namely that equation (2) provides a short cut to the direct determination of a complete orthonormal set of time-dependent solutions of (1). While with t -independent H the choice between a time-dependent and the usual stationary basis is just a matter of taste, the former can be more conveniently generalised for the case of t -dependent fields.

3. Electron in an oscillating field

In the following we shall consider the interaction of a free electron with a periodic external field. The corresponding Schrödinger equation with a scalar potential reads

$$\left(\hat{\mathbf{p}}^2/2m - e\mathbf{E}_0 \hat{\mathbf{x}} \cos \omega t\right)\psi = i\hbar \partial\psi/\partial t, \quad \mathbf{A} = 0, \quad (4)$$

and with a vector potential

$$\frac{1}{2m} \left(\hat{\mathbf{p}} + \frac{e\mathbf{E}_0}{\omega} \sin \omega t \right)^2 \psi' = i\hbar \frac{\partial \psi'}{\partial t}, \quad (4a)$$

$$\mathbf{A}' = -(c/\omega)\mathbf{E}_0 \sin \omega t, \quad \Phi' = 0.$$

The gauge transformation described above is effected in this case by $\chi = -(c/\omega)\mathbf{E}_0\mathbf{x} \sin \omega t$. Equation (4a) in the momentum representation is readily integrable again and its general solution is

$$\psi'(\mathbf{p}, t) = g(\mathbf{p}) \exp\left[-\frac{i}{\hbar} \frac{1}{2m} \int_0^t \left(\mathbf{p} + \frac{e\mathbf{E}_0}{\omega} \sin \omega\tau\right)^2 d\tau\right]. \quad (5)$$

In the limit $\omega \rightarrow 0$ this solution coincides with (2), and $g(\mathbf{p})$ satisfies the conditions stated above for $f(\mathbf{p})$. The solution of (4) is given by $\psi(\mathbf{p}, t) = \psi'[\mathbf{p} - (e\mathbf{E}_0/\omega) \sin \omega t]$, and the corresponding solution in the coordinate representation is

$$\psi'(\mathbf{x}, t) = \frac{1}{(2\pi\hbar)^{1/2}} \int d^3\mathbf{p}' g(\mathbf{p}') \exp\left\{\frac{i}{\hbar} \left[\mathbf{p}'\mathbf{x} - \frac{1}{2m} \int_0^t \left(\mathbf{p}' + \frac{e\mathbf{E}_0}{\omega} \sin \omega\tau\right)^2 d\tau\right]\right\} \quad (5a)$$

and for $\psi(\mathbf{x}, t)$ we have

$$\psi(\mathbf{x}, t) = \exp\left(\frac{i}{\hbar} \frac{e\mathbf{E}_0}{\omega} \mathbf{x} \sin \omega t\right) \psi'(\mathbf{x}, t).$$

The usual plane wave solution given by Keldysh (1965) can be obtained by substituting $g(\mathbf{p}') = \delta(\mathbf{p} - \mathbf{p}')$. The meaning of this solution becomes clearer if we consider the time evolution operator $\hat{U}(t)$ of equation (4a). From the definition of the evolution operator we have

$$\hat{U}(t) = \exp\left[-\frac{i}{\hbar} \int_0^t \left(\hat{\mathbf{p}} + \frac{e}{\omega} \mathbf{E}_0 \sin \omega\tau\right)^2 d\tau\right], \quad \hat{U}(0) = \hat{1}. \quad (6)$$

When one applies it to a momentum state $|\mathbf{p}\rangle$ (with the bra and ket vector notation), one obtains the time development of the state. As $\hat{U}(t)$ is diagonal in the \mathbf{p} representation, its only effect on $|\mathbf{p}\rangle$ is to multiply it by a complex c number which is a function of \mathbf{p} and t . Furthermore the modulus of this number is unity, which is a consequence of the unitary nature of $\hat{U}(t)$:

$$\hat{U}(t)|\mathbf{p}\rangle = U(\mathbf{p}, t)|\mathbf{p}\rangle, \quad |U(\mathbf{p}, t)|^2 = 1. \quad (7)$$

Here $U(\mathbf{p}, t)$ is the matrix element of the time evolution operator and from (6) it can be seen that it has the form predicted by the Floquet theorem for the solution of differential equations with periodic coefficients (Shirley 1965). From (7) we see that a given momentum state remains always the same, only its phase will change in time. The same result can be expressed using a somewhat different language. The equation of motion for the matrix elements of the density operator $\hat{\rho}$ in the momentum representation is, from either (1a) or (4a),

$$\begin{aligned} i\hbar \frac{\partial \langle \mathbf{p}_1 | \hat{\rho} | \mathbf{p}_2 \rangle}{\partial t} &= i\hbar \frac{\partial \rho(\mathbf{p}_1, \mathbf{p}_2, t)}{\partial t} = H(\mathbf{p}_1)\rho(\mathbf{p}_1, \mathbf{p}_2, t) - H(\mathbf{p}_2)\rho(\mathbf{p}_1, \mathbf{p}_2, t) \\ &= \left[\frac{1}{2m} (\mathbf{p}_1^2 - \mathbf{p}_2^2) - \frac{e}{mc} \mathbf{A}(t)(\mathbf{p}_1 - \mathbf{p}_2) \right] \rho(\mathbf{p}_1, \mathbf{p}_2, t). \end{aligned} \quad (8)$$

In either case, the Hamiltonian H is diagonal in the \mathbf{p} -representation and $H(\mathbf{p}_1)$ and $H(\mathbf{p}_2)$ are the corresponding eigenvalues. The solution is again easily obtained by direct integration:

$$\rho(\mathbf{p}_1, \mathbf{p}_2, t) = \rho(\mathbf{p}_1, \mathbf{p}_2, 0) \exp\left[-\frac{i}{\hbar} \int_0^t \left(\frac{1}{2m} (\mathbf{p}_1^2 - \mathbf{p}_2^2) - \frac{e}{mc} \mathbf{A}(\tau)(\mathbf{p}_1 - \mathbf{p}_2) \right) d\tau\right]. \quad (9)$$

By setting $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$ we see that the initial momentum distribution function (given by the diagonal elements of $\hat{\rho}$ at $t = 0$) remains unchanged:

$$\rho(\mathbf{p}, \mathbf{p}, t) = \rho(\mathbf{p}, \mathbf{p}, 0). \quad (10)$$

From (10) we may conclude, in agreement with (7), that the momentum distribution of a free electron in an external field remains unchanged in the dipole approximation. Deviation from this result is expected only if the dipole approximation is dropped.

4. Application to nonlinear direct and inverse bremsstrahlung

Consider now the problem of the interaction of an electron with an external field in the presence of a background potential. This corresponds to modelling the induced bremsstrahlung process. The Schrödinger equation is

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{1}{2m} \left(\hat{\mathbf{p}} + \frac{e\mathbf{E}_0}{\omega} \sin \omega t \right)^2 + V(\mathbf{r}) \right] \psi. \quad (11)$$

Depending on the nature of the problem, different approximation methods for the solution of equation (11) are worked out. If the field is of low intensity the usual perturbation theory applies (Gontier and Trahin 1971). If, however, the external field is of the same (or higher) order of magnitude than the static field given by $V(\mathbf{r})$, then other methods would be necessary. The other limiting case is when the external field is so strong that the background potential $V(\mathbf{r})$ can be treated as a perturbation. It seems to be quite natural, at least in scattering problems which frequently occur in highly ionised plasma, to build up a perturbation series in powers of $V(\mathbf{r})$, where the complete set of the plane wave solutions of (4a) is used as a basis. Let us denote $\psi'(\mathbf{x}, t)$ of (5a) in the case of $g(\mathbf{p}') = \delta(\mathbf{p} - \mathbf{p}')$ by ψ_p , and look for the solution of equation (11) in the following form:

$$\psi_i(\mathbf{x}, t) = \psi_{p_i} + \psi_1, \quad \psi_1 = \int a_p(t) \psi_p d^3 p \quad (12)$$

with the initial condition $\psi_i(t = 0) = \psi_{p_i}$. By substituting into (11) and neglecting terms higher than first order in $V(\mathbf{r})$, one obtains the following differential equation for $a_p(t)$:

$$i\hbar \frac{da_p(t)}{dt} = V(\mathbf{p} - \mathbf{p}_i) \exp \left\{ \frac{i}{\hbar} \frac{1}{2m} \int_0^t \left[\left(\mathbf{p} + \frac{e\mathbf{E}_0}{\omega} \sin \omega \tau \right)^2 - \left(\mathbf{p}_i + \frac{e\mathbf{E}_0}{\omega} \sin \omega \tau \right)^2 \right] d\tau \right\}. \quad (13)$$

The solution of this equation is simple, and from (12) we have an explicit expression for $\psi_1(\mathbf{x}, t)$ in the first approximation. From this latter expression the transition matrix element for the scattering process has the form

$$\begin{aligned} T_{fi} &= (\psi_{p_f}, \psi_1) \\ &= -(i/\hbar) V(\mathbf{p}_f - \mathbf{p}_i) \\ &\quad \times \int_0^t \exp \left\{ \frac{i}{\hbar} \frac{1}{2m} \int_0^{t'} \left[\left(\mathbf{p}_f + \frac{e\mathbf{E}_0}{\omega} \sin \omega \tau \right)^2 - \left(\mathbf{p}_i + \frac{e\mathbf{E}_0}{\omega} \sin \omega \tau \right)^2 \right] d\tau \right\} dt' \end{aligned} \quad (14)$$

where ψ_{p_f} is the final-state plane wave function, satisfying the free particle equation

(4a). The term $-(i/\hbar)V(\mathbf{p}_f - \mathbf{p}_i)$ is the usual scattering amplitude in the Born approximation. The structure of the second factor on the right-hand side satisfies the general requirements found for the case of a Hamiltonian which is periodic in time (Reiss 1970, Bergou 1975). The obvious advantage of the present method is that (14) naturally emerges as the first term in a perturbation expansion in powers of V while in Reiss's method or in the different approach by Faisal (1973), Gontier and Rahman (1974) and Bergou (1976), it was not quite clear how to obtain further corrections to the result. The use of the basis set (5) implies that terms containing the vector potential \mathbf{A} are not regarded as small perturbations, and thereby we avoid some objections (Cohen-Tannoudji *et al* 1973) raised against the much discussed method of Reiss.

Using the definition of the $J_n(z)$ Bessel functions, the periodically time dependent part of the exponential term in the integrand can be expanded into power series of the absorbed and emitted photons:

$$\exp\left(\frac{i}{\hbar}z \cos \omega t\right) = \sum_{n=-\infty}^{\infty} i^n J_n(z) \exp\left(\frac{i}{\hbar}n \hbar \omega t\right), \quad z = \frac{eE_0}{m\hbar\omega^2} \mathbf{Q},$$

where

$$\mathbf{Q} = \mathbf{p}_i - \mathbf{p}_f \quad \text{and} \quad \mathbf{p}_i^2/2m = \mathbf{p}_f^2/2m - n \hbar \omega.$$

\mathbf{p}_i and \mathbf{p}_f are the initial and final momenta, respectively, and so \mathbf{Q} accounts for the momentum change in the scattering. If this expansion is introduced into (14) the time integration can easily be carried out and one obtains the following final result for the scattering cross section:

$$\frac{d\sigma}{d\Omega} = \sum_{n=-\infty}^{\infty} \frac{d\sigma^{(n)}}{d\Omega}, \quad \frac{d\sigma^{(n)}}{d\Omega} = \frac{p_f}{p_i} J_n^2(z) \frac{d\sigma_{\text{Born}}^{(el)}}{d\Omega}. \quad (15)$$

Here $d\sigma_{\text{Born}}^{(el)}/d\Omega$ is the differential cross section of the elastic scattering on a $V(\mathbf{r})$ background potential in the Born approximation, and the Bessel function of order n accounts for the modification of it due to n -photon processes. The scattering process is elastic with respect to the background potential but inelastic with respect to the external field. Equation (15) is formally similar to the result found by Reiss (1970) for bound-free transitions, but even for the same problem, the two approaches should give different coefficients, due to our non-perturbational treatment of \mathbf{A} . It should be noted that our method gives systematic expansion in powers of V but not in photon number n : going to second order in V would alter all terms in (15).

5. Summary

The result (15) was obtained earlier by several authors, however by a somewhat different method, the so-called space translation transformation (Faisal 1973, Gontier and Rahman 1974, Bergou 1976). The main advantage of the method outlined in the present paper is the simplicity of obtaining higher-order approximations in $V(\mathbf{r})$ (the background potential) in contrast to the space translation method, where the perturbation potential is much more complicated and therefore the extension of the Born approximation is difficult. The generalisation of this method to relativistic electrons as well as its extension beyond the dipole approximation is treated in Bergou and Varró (1980).

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